FAMILIES OF VALUATIONS AND SEMIGROUPS OF FRACTIONARY IDEAL CLASSES

BY

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Introduction. Let R be an integral domain with quotient field K. For any valuation v on K which is nonnegative on R, we let $P(v)=\{x \in R \mid v(x)>0\}$. $P(v)$ is a prime ideal of R and is called the center of v on R. In this paper we are concerned mainly with integral domains R which satisfy the following: There exists a family F of valuations on K such that

(i) Each $v \in F$ has rank one.

(ii) $R=\bigcap_{v\in F} R_v$.

(iii) $R_v=R_{P(v)}$, for each $v \in F$.

A family F of valuations on K is said to be of finite character if for $x \in K$, $x\neq 0$, there are only a finite number of $v \in F$ such that $v(x)\neq 0$. R is called a Krull domain if there is a family F of finite character satisfying (i), (ii), (iii), with the additional requirement that each $v \in F$ be discrete. R is called an almost-Krull (AK) domain [7] if R_P is a Krull domain for every proper nonzero prime P of R. It follows that R is almost Dedekind (AD) iff R is an AK-domain in which proper prime ideals are maximal [7].

Using the family F of valuations we construct a partially ordered semigroup $\mathscr{A}(R)$ of fractionary ideal classes in §1 and study the relation between $\mathscr{A}(R)$ and $\mathscr{D}(R)$, the divisor group of R (see [1]). Necessary and sufficient conditions for $\mathscr{A}(R)$ and $\mathscr{D}(R)$ to be isomorphic are determined. In §2, condition (S) of [3] is studied. §3 consists of an example.

The notation concerning $\mathscr{D}(R)$ is that of [1]. Otherwise, the notation of [8] is used. Prime ideals are always nonzero and not all of R.

1. In order to make this paper as self contained as possible we first list the necessary background results from [1]. R will denote a commutative integral domain with identity and quotient field K. $I(R)$ will denote the collection of nonzero fractionary ideals of R. A fractionary ideal of the form Rx, $x \in K$, $x\neq 0$, is called a principal fractionary ideal.

A relation $\prec$ is defined on $I(R)$ as follows: $A\prec B$ iff every principal fractionary ideal of R which contains A also contains B. The relation $\prec$ is a preorder on $I(R)$; i.e., $\prec$ is a symmetric, transitive relation. If we define $\equiv$ on $I(R)$ by $A\equiv B$ iff $A\prec B$ and $B\prec A$, then $\equiv$ is an equivalence relation on $I(R)$. For $A \in I(R)$, $\operatorname{div}_R(A)$

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denotes the equivalence class of A with respect to $\equiv$ and is called the divisor of A; $\mathscr{D}(R)$ denotes the set of all such equivalence classes.

For $A \in I(R)$, we put $\tilde{A}=\bigcap_{A\subseteq Rx} Rx$. A fractionary ideal B of R is said to be divisoriel if $B=\tilde{B}$. It follows that for $A \in I(R)$, $\mathrm{div}_R(A)=\mathrm{div}_R(\tilde{A})$ and that $\tilde{A}$ is the unique divisoriel fractionary ideal belonging to $\mathrm{div}_R(A)$. It also follows from the definition that $(\tilde{A}\tilde{B})^{\sim}=(AB)^{\sim}$ for $A, B \in I(R)$ so that $\mathscr{D}(R)$ together with the operation $+$, defined by $\mathrm{div}_R(A)+\mathrm{div}_R(B)=\mathrm{div}_R(AB)$, is a commutative semigroup with identity $0=\mathrm{div}_R(R)$. If we define $\leqq$ on $\mathscr{D}(R)$ by $\mathrm{div}_R(A)\leqq\mathrm{div}_R(B)$ iff $A\prec B$ then $\mathscr{D}(R)$ is a lattice ordered semigroup with respect to the partial ordering $\leqq$. Furthermore, $\mathscr{D}(R)$ is a group iff R is completely integrally closed [1, p. 5, Theorem 1].

Let F be a family of valuations on K with the following properties:

(i) Each $v \in F$ has rank one.

(ii) $R=\bigcap_{v\in F} R_v$.

(iii) For each $v \in F$, $R_v=R_{P(v)}$, where $P(v)$ denotes the center of v on R.

Occasionally in place of (i) we shall substitute

(i′) Each $v \in F$ has rank one and is discrete.

DEFINITION 1.1. For $v \in F$, $A \in I(R)$, put $v(A)=\inf\{v(a) \mid a \in A\}$.

LEMMA 1.2. *If $A, B \in I(R)$, $v \in F$, then $v(AB)=v(A)+v(B)$.*

Proof. See [4, p. 712], Theorem 1, part (2).

Now, for $A, B \in I(R)$, define $A\sim B$ iff $v(A)=v(B)$ for all $v \in F$. Then $\sim$ is an equivalence relation on $I(R)$. For $A \in I(R)$ we let $[A]$ denote the equivalence class of A with respect to $\sim$, and we let $\mathscr{A}(R)$ denote the set of all such equivalence classes.

Define $+$ on $\mathscr{A}(R)$ by $[A]+[B]=[AB]$. Then $+$ is well defined. Since multiplication of fractionary ideals is commutative and associative, $\mathscr{A}(R)$ together with $+$ is a commutative semigroup with identity $0=[R]$.

LEMMA 1.3. *If $A=Rx$ is a principal fractionary ideal, then $v(A)=v(x)$ for all $v \in F$.*

Proof. $v(A)=v(Rx)=\inf_{rx\in Rx} v(rx)=\inf_{r\in R} v(r)+v(x)=v(1)+v(x)=v(x)$.

If G is a group and I is any nonempty index set, we let G^I denote the direct product of I copies of G and we let $G^{(I)}$ denote the direct sum of I copies of G. We shall assume that the value group of each $v \in F$ is a subgroup of the additive group of real numbers. When $v \in F$ is discrete we assume, without loss of generality, that the value group of v is the additive group of integers. X denotes the real numbers and Z denotes the integers.

PROPOSITION 1.4. *Let $F=\{v_i \mid i \in I\}$ where I is an index set. The map $f: \mathscr{A}(R)\to X^I$, defined by $f([A])=(v_i(A))_{i\in I}$, is a monomorphism.*

Proof. The proof is straightforward and is omitted.

It follows from Proposition 1.4 that $\mathscr{A}(R)$ is a semigroup in which the cancellation law holds.

We now introduce a partial ordering for $\mathscr{A}(R)$.

DEFINITION 1.5. For $[A]$, $[B] \in \mathscr{A}(R)$, put $[A] \leqq [B]$ iff $v(A) \leqq v(B)$ for all $v \in F$.

PROPOSITION 1.6. *$\mathscr{A}(R)$ is partially ordered by $\leqq$.*

Proof. The proof is straightforward and is omitted.

As usual, if $[A]$, $[B] \in \mathscr{A}(R)$ are such that $[A] \leqq [B]$ and $[A] \neq [B]$, we write $[A] < [B]$. Since $[R]=0 \in \mathscr{A}(R)$, $[A] \in \mathscr{A}(R)$ is such that $[A] \geqq 0$ iff A is an ideal of R. For if A is an ideal of R, then $A \subseteq R$ so that $v(A) \geqq v(R)=0$, for all $v \in F$. On the other hand, if $[A] \geqq 0$, then $v(A) \geqq 0$ for all $v \in F$ so that $A \subseteq \bigcap_{v \in F} R_v = R$. Furthermore, if each $v \in F$ is discrete, then $[A] > 0$ iff $A \subseteq P(v)$ for some $v \in F$. For if $[A] > 0$, then, since each v is discrete, $v(A) \geqq 1 > 0$ for some $v \in F$. But then $A \subseteq P(v)$, and conversely. We can use these properties of $\leqq$ to characterize the positive elements of $\mathscr{A}(R)$ when the elements of F are discrete.

For n a positive integer and P a minimal prime of R, put $P^{(n)} = P^n R_P \cap R$. We shall assume that F satisfies (i′), (ii), (iii) in Propositions 1.7 and 1.8.

PROPOSITION 1.7. *If P is a minimal prime of R and $P=P(v)$ for some $v \in F$, then $P^{(n)} = \{x \in R \mid v(x) \geqq n\}$ for every positive integer n.*

Proof. We have $P^{(n)} = P^n R_P \cap R = (PR_P)^n \cap R$. So if $x \in P^{(n)}$, then $x \in (PR_P)^n$ and so $v(x) \geqq n$; i.e., $P^{(n)} \subseteq \{x \in R \mid v(x) \geqq n\}$. On the other hand, if $x \in R$ is such that $v(x) \geqq n$, then $x \in P$ and hence $x \in PR_P$. Since $v(x) \geqq n$ we have $x \in (PR_P)^n$, and so $x \in P^{(n)}$; i.e., $\{x \mid x \in R, v(x) \geqq n\} \subseteq P^{(n)}$.

As is well known, $P^{(n)}$ is a P-primary ideal of R.

PROPOSITION 1.8. *Let A be an ideal of R such that* $[A] > 0$, *and let*

$$J = \{j \in I \mid v_j(A) > 0\}.$$

Then $A \subseteq \bigcap_{j \in J} P_j^{(n_j)}$, and $[A] = [\bigcap_{j \in J} P_j^{(n_j)}]$, where for each $j \in J$, $P_j = P(v_j)$ and $n_j = v_j(A)$.

Proof. If $x \in A$, $j \in J$, then $v_j(x) \geqq v_j(A) = n_j$; i.e., $x \in P_j^{(n_j)}$ by Proposition 1.7. Thus for each $j \in J$, $A \subseteq P_j^{(n_j)}$; i.e., $A \subseteq \bigcap_{j \in J} P_j^{(n_j)}$; which proves the first assertion. Now let $k \in J$. Since $A \subseteq \bigcap_{j \in J} P_j^{(n_j)}$, we have $v_k(A) \geqq v_k(\bigcap_{j \in J} P_j^{(n_j)})$. On the other hand, let $x \in \bigcap_{j \in J} P_j^{(n_j)}$. Then $x \in P_k^{(n_k)}$, and $v_k(x) \geqq n_k = v_k(A)$; i.e., $v_k(\bigcap_{j \in J} P_j^{(n_j)}) \geqq v_k(A)$. So if $k \in J$, then $v_k(A) = v_k(\bigcap_{j \in J} P_j^{(n_j)})$. If $k \in I-J$, then $0 = v_k(A) \geqq v_k(\bigcap_{j \in J} P_j^{(n_j)}) \geqq 0$; i.e., $v_k(A) = 0 = v_k(\bigcap_{j \in J} P_j^{(n_j)})$ for all $k \in I-J$, Thus

$$v_i(A) = v_i\Big(\bigcap_{j \in J} P_j^{(n_j)}\Big)$$

for all $i \in I$; i.e., $[A] = [\bigcap_{j \in J} P_j^{(n_j)}]$.

It follows that $\bigcap_{j \in J} P_j^{(n_j)}$ is the largest ideal B of R such that $[A]=[B]$.

We now drop the assumption that each $v \in F$ is discrete so that F satisfies (i), (ii) and (iii). Property (i) says that R_v is a rank one valuation ring and hence completely integrally closed for each $v \in F$. Property (ii) shows that R is the intersection of completely integrally closed overrings and hence is completely integrally closed. So (i) and (ii) insure that $\mathscr{D}(R)$ is a group. We now study relations between the semigroup $\mathscr{A}(R)$ and the group $\mathscr{D}(R)$.

The next two propositions have been proved in [6] for the case when F is the family of essential valuations of an AD-domain R.

Proposition 1.9. *Let $A \in I(R)$. Then, considering $[A]$ and $\operatorname{div}_R(A)$ as subsets of $I(R)$, $[A] \subseteq \operatorname{div}_R(A)$.*

Proof. Let $B \in [A]$. Then $v(B)=v(A)$ for all $v \in F$. If $A \subseteq Rx$, then $v(A) \geqq v(Rx) = v(x)$, and so $v(B) \geqq v(x)$ for all $v \in F$. If $b \in B$, then $v(b)-v(x) \geqq 0$ for all $v \in F$; i.e., $v(b/x) \geqq 0$ for all $v \in F$. Thus if $b \in B$ then $b/x \in \bigcap_{v \in F} R_v = R$, and $b \in Rx$; i.e., $B \subseteq Rx$. Similarly, if $B \subseteq Ry$ then $A \subseteq Ry$. In this case, $\tilde{A} = \bigcap_{A \subseteq Ry} Ry = \bigcap_{B \subseteq Rx} Rx = \tilde{B}$, and $\operatorname{div}_R(A) = \operatorname{div}_R(B)$. Hence $B \in \operatorname{div}_R(A)$.

Proposition 1.10. *The map $g: \mathscr{A}(R) \to \mathscr{D}(R)$ defined by $g([A]) = \operatorname{div}_R(A)$ is an order preserving homomorphism of the partially ordered semigroup $\mathscr{A}(R)$ onto the lattice ordered group $\mathscr{D}(R)$.*

Proof. Proposition 1.9 shows that g is well defined and onto. It follows directly that g is a homomorphism. To see that g preserves order, suppose $[A], [B] \in \mathscr{A}(R)$ with $[A] \leqq [B]$. If $A \subseteq Rx$, it follows as in the proof of 1.9 that $B \subseteq Rx$ so that $A \subseteq B$, and hence $\operatorname{div}_R(A) \leqq \operatorname{div}_R(B)$.

Now let T be a domain such that $R \subseteq T \subseteq K$ and such that there is a subfamily G of F such that $T = \bigcap_{w \in G} R_w$. It is easy to show that G is a family of valuations for T satisfying (i), (ii), (iii).

Proposition 1.11. *The map $\sigma: \mathscr{A}(R) \to \mathscr{A}(T)$, defined by $\sigma([A]) = [AT]$, is an order preserving homomorphism of $\mathscr{A}(R)$ onto $\mathscr{A}(T)$.*

Proof. Here, $\mathscr{A}(T)$ denotes the semigroup of fractionary ideal classes of T formed with the family G.

It is clear that σ is well defined. To see that σ is onto, let $\mathscr{U}$ be any nonzero fractionary ideal of T. Then $\mathscr{U} = (1/d)\mathscr{B}$, where $\mathscr{B}$ is an ideal of T, $d \in R$, $d \neq 0$. Put $A = (1/d)B$, where $B = \mathscr{U} \cap R$. It can be shown that $v(B) = v(\mathscr{B})$ for all $v \in G$, and hence $\sigma([A]) = [\mathscr{U}]$. It is straightforward to show that σ is a homomorphism which preserves order.

Corollary 1.12. *If T is as in 1.11, then $\mathscr{D}(T)$ is a homomorphic image of $\mathscr{A}(R)$.*

Let T be as in 1.11, and consider the following diagram:

Diagram 1.13.

$$\begin{tikzcd} \mathscr{A}(R) \arrow[r, "\sigma"] \arrow[d, "g_1"'] & \mathscr{A}(T) \arrow[d, "g_2"] \\ \mathscr{D}(R) \arrow[r, dashed, "\rho"] & \mathscr{D}(T) \end{tikzcd}$$

Here σ is the homomorphism of 1.11, g_1 and g_2 are the canonical homomorphisms of 1.10. In general, this diagram may not be completed commutatively by a homomorphism ρ. For let R be an AD-domain which is not Dedekind, and let F denote the family of essential valuations of R. By a result in [6], R contains at least one proper prime P which is not divisoriel. Then $P<\tilde{P}$, and hence $\tilde{P}=R$ since P is maximal. Since R is AD, there is $v\in F$ such that $P=P(v)$, for some $v\in F$. Take $T=R_P=R_v$ and assume that ρ completes the following diagram commutatively:

Diagram 1.14.

$$\begin{tikzcd} \mathscr{A}(R) \arrow[r, "\sigma"] \arrow[d, "g_1"'] & \mathscr{A}(R_P) \arrow[d, "g_2"] \\ \mathscr{D}(R) \arrow[r, "\rho"] & \mathscr{D}(R_P) \end{tikzcd}$$

Then we must have $\rho(g_1([P]))=g_2(\sigma([P]))$. However, $g_1([P])=\operatorname{div}_R(P)=0$ (since $\tilde{P}=R$) so that $\rho(g_1([P]))=0$; and on the other hand $\sigma([P])=[PR_P]$. But since R_P is a Dedekind domain with unique maximal ideal PR_P, we have that $\operatorname{div}_{R_P}(PR_P)>0$; i.e., $g_2(\sigma([P]))=\operatorname{div}_{R_P}(PR_P)>0$. Thus $\rho g_1\neq g_2\sigma$, contradicting our assumption on ρ. This proves the assertion that, in general, Diagram 1.13 may not be completed commutatively.

Equivalent conditions for an AD-domain R to be Dedekind are given in terms of $\mathscr{A}(R)$ in [6]. If we are to extend these results we need to know something about the inverses of elements of $\mathscr{A}(R)$ whenever they exist.

PROPOSITION 1.15. *If* $[A]\in\mathscr{A}(R)$ *has an inverse then* $-[A]=[R:A]$.

Proof. Suppose $[A]\in\mathscr{A}(R)$ has an inverse $[B]$. Since the canonical map g: $\mathscr{A}(R)\to\mathscr{D}(R)$ is a homomorphism, we must have that $g(-[A])=-g([A])=-\operatorname{div}_R(A)=\operatorname{div}_R(R:A)$. Thus $g([B])=g(-[A])=\operatorname{div}_R(R:A)$. But by definition of g, $g([B])=\operatorname{div}_R(B)$ so that $\operatorname{div}_R(B)=\operatorname{div}_R(R:A)$. Since $R:A$ is divisoriel we have $B\subseteq\tilde{B}=R:A=R:\tilde{A}$. Then $AB\subseteq A(R:A)\subseteq R$ so that $0=[AB]\geqq[A(R:A)]\geqq 0$. Thus $0=[A]+[B]=[A]+[R:A]$; i.e., $-[A]=[B]=[R:A]$.

Now consider the following diagram.

Diagram 1.16.

$$\begin{tikzcd} \mathscr{A}(R) \arrow[r, "f"] \arrow[d, "g"'] & X^I \\ \mathscr{D}(R) \arrow[ur, dashed, "\lambda"'] & \end{tikzcd}$$

Here g is the canonical homomorphism and f is the homomorphism of 1.4. g is surjective and f is injective.

PROPOSITION 1.17. *Diagram* 1.16 *may be completed commutatively by a homomorphism* λ *iff* g *is an isomorphism.*

We can now prove the following theorem.

THEOREM 1.18. *Let R be an integral domain with quotient field K, and let F be a family of valuations satisfying* (i), (ii), (iii). *The following statements are equivalent.*

(1) $\mathscr{A}(R)$ *is a group.*

(2) $R:A=R:B \Rightarrow [A]=[B]$, *for all* $A, B \in I(R)$.

(3) $v(A)=v(\tilde{A})$ *for all* $A \in I(R)$ *and* $v \in F$.

(4) *The map* $g: \mathscr{A}(R) \rightarrow \mathscr{D}(R)$ *is an isomorphism.*

Proof. (1) $\Rightarrow$ (2) Suppose $\mathscr{A}(R)$ is a group. Let $A, B \in I(R)$ be such that $R:A=R:B$. By 1.15 we have $-[A]=[R:A]=[R:B]=-[B]$ and hence $[A]=[B]$.

(2) $\Rightarrow$ (3) We have $R:A=R:\tilde{A}$ for all $A \in I(R)$. If (2) holds then $[A]=[\tilde{A}]$ for all $A \in I(R)$; i.e., $v(A)=v(\tilde{A})$ for all $v \in F$, $A \in I(R)$.

(3) $\Rightarrow$ (4) Consider Diagram 1.16. If $v(A)=v(\tilde{A})$ for all $v \in F$ and $A \in I(R)$, we can define $\lambda: \mathscr{D}(R) \rightarrow X^I$ by $\lambda(\operatorname{div}_R(A))=(v_i(A))_{i\in I}$. It follows that λ is a homomorphism and that $\lambda \circ g=f$. By 1.17, g is an isomorphism.

(4) $\Rightarrow$ (1) obvious.

We observe that the converse of statement (2) in 1.18 is always true in R. For if $[A]=[B]$, then $A \in [B] \subseteq \operatorname{div}_R(B)$, and $B \in \operatorname{div}_R(B)$ so that $\operatorname{div}_R(A)=\operatorname{div}_R(B)$ and hence $R:A=R:B$.

When the valuations in F are discrete we obtain a partial generalization of a result in [6] with the aid of the following lemma.

LEMMA 1.19. *Assume that each $v \in F$ is discrete. Then for each $v \in F$, if* $\operatorname{div}_R(P(v)) \neq 0$ *then* $P(v)=(P(v))^{\sim}$.

Proof. We have $P(v) \subseteq (P(v))^{\sim} \subseteq R$. If $P(v)<(P(v))^{\sim}$, there is $x \in (P(v))^{\sim}$, $x \notin P(v)$. Then $v(x)=0=v(P(v))^{\sim}$. Also, for $w \in F$, $w \neq v$, we have $0=w(P(v)) \geqq w(P(v))^{\sim} \geqq 0$. Thus $[(P(v))^{\sim}]=0$. Since the canonical map g from $\mathscr{A}(R)$ onto $\mathscr{D}(R)$ is a homomorphism, we should have $0=[(P(v))^{\sim}] \rightarrow \operatorname{div}_R(P(v))^{\sim}=\operatorname{div}_R(P(v))=0$. But $\operatorname{div}_R(P(v)) \neq 0$ by assumption. Thus we must have $(P(v))^{\sim}=P(v)$ if $\operatorname{div}_R(P(v)) \neq 0$.

THEOREM 1.20. *Assume each $v \in F$ is discrete. Then the canonical map g from $\mathscr{A}(R)$ onto $\mathscr{D}(R)$ is an isomorphism iff $P(v)$ is divisoriel for each $v \in F$.*

Proof. ($\Rightarrow$) Suppose g is an isomorphism. If $P=P(v)$ for some $v\in F$, then $[P]>0$ since $v(P)=1$. If P is not divisoriel then $g([P])=\operatorname{div}_R(P)=0$, by Lemma 1.19. But then g is not 1-1 and hence not an isomorphism. For $[R]=0\neq[P]$.

($\Leftarrow$) Suppose $P(v)$ is divisoriel for each $v\in F$. Then if $A\in I(R)$ is such that $\operatorname{div}_R(A)=0$ we must have $A\subseteq R$ (for this result see [1, bottom of p. 4]). Moreover, $A\nsubseteq P(v)$ for any $v\in F$. For if $A\subseteq P=P(v)$ for some $v\in F$, then $\operatorname{div}_R(A)\geqq\operatorname{div}_R(P)>0$, a contradiction. Thus $g([A])=0$ iff $[A]=0$. Now suppose $[A]$, $[B]\in\mathscr{A}(R)$ are such that $g([A])=g([B])$. Then $\operatorname{div}_R(A)=\operatorname{div}_R(B)$ so that $\operatorname{div}_R(A)-\operatorname{div}_R(B)=0=\operatorname{div}_R(B)-\operatorname{div}_R(A)$; i.e., $\operatorname{div}_R(A:B)=0=\operatorname{div}_R(B:A)$. Since $g([A:B])=g([B:A])=0$, we must have $[A:B]=0=[B:A]$. Since each $v\in F$ is discrete, for each $v\in F$ there is $x\in A:B$ such that $v(x)=v(A:B)=0$. Now $xB\subseteq A$ (by definition of $A:B$) so that $v(x)+v(B)=v(xB)\geqq v(A)$; i.e., $v(B)\geqq v(A)$. Thus $v(B)\geqq v(A)$ for all $v\in F$. Similarly $v(A)\geqq v(B)$ for all $v\in F$, and $[A]=[B]$. This shows that g is 1-1 and hence an isomorphism.

When R is AD, the author has shown in [6] that $P(v)$ is divisoriel for each $v\in F$ iff R is Dedekind. To date, however, the author has been unable to prove the following conjecture: If R is AK and $P(v)$ is divisoriel for each $v\in F$, then R is a Krull domain.

When R is AK, we do have the following theorem.

THEOREM 1.21. *Let R be an* AK*-domain with family F of essential valuations and let Δ denote the collection of maximal ideals of R. Every minimal prime of R is divisoriel iff $\tilde{A}=\bigcap_{M\in\Delta}(AR_M)^{\sim}$ for every ideal A of R.*

Proof. Here $\tilde{A}=\bigcap_{A\subseteq Rx}Rx$ and $(AR_M)^{\sim}=\bigcap_{AR_M\subseteq R_My}R_My$. For any maximal ideal M of R, F_M denotes the family of essential valuations of the Krull domain R_M. Recall that $F_M\subseteq F$.

($\Rightarrow$) Let A be an ideal of R. If M is any maximal ideal of R then $v(A)=v(AR_M)$ for all $v\in F_M$. Since R_M is a Krull domain, $v(AR_M)=v(AR_M)^{\sim}$ for all $v\in F_M$ so that $v(A)=V(AR_M)^{\sim}$ for all $v\in F_M$.

Case 1. $v(A)=0$ for all $v\in F$.

Then $P<A$ for every minimal prime P of R. In this case $\tilde{A}=R=\bigcap_{M\in\Delta}R_M=\bigcap_{M\in\Delta}(AR_M)^{\sim}$.

Case 2. $v(A)>0$ for some $v\in F$.

For each maximal ideal M of R, if there is $v\in F_M$ such that $0<v(A)=v(AR_M)^{\sim}$, then we can write

$$(AR_M)^{\sim} = \bigcap_{v_i\in F_M;v_i(A)>0} Q_i^{(n_i)},$$

where $n_i=v_i(AR_M)^{\sim}$ and Q_i is the center of v_i on R_M. Then for each i such that $v_i(A)>0$ we have $Q_i=P_iR_i$ where $P_i=P(v_i)$ in R. Thus $(AR_M)^{\sim}=\bigcap_i(P_iR_M)^{(n_i)}=\bigcap_i((P_iR_M)^{n_i}R_{P_i})\cap R_M=\bigcap_i(P_i^{n_i}R_{P_i}\cap R_M)$ where i runs over all indices such that $v_i\in F_M$ and $v_i(A)>0$, and $n_i=v_i(A)$ for each such i. It can then be shown that, $C=\bigcap_{M\in\Delta}(AR_M)^{\sim}=\bigcap\{P_i^{(n_i)}\mid v_i\in F, v_i(A)>0\}$. Then $[C]=[A]$. Since $[A]=\operatorname{div}_R(A)$, it follows that $\tilde{A}=C$.

($\Leftarrow$) Suppose P is a minimal prime of R. If $M \in \Delta$, then either $P \subseteq M$ or $P \nsubseteq M$. If $P \nsubseteq M$ then $PR_M = R_M$. If $P \subseteq M$, then PR_M is a minimal prime of the Krull domain R_M and thus $(PR_M)^{\sim} = PR_M$. Since P is contained in some maximal ideal M we have $\tilde{P} = \bigcap_{M \in \Delta} (PR_M)^{\sim} = \bigcap_{M \in \Delta} PR_M = P$.

We now drop the assumption that R is an AK-domain and assume only that F satisfies axioms (i), (ii), (iii) at the beginning of this section. The next lemma tells us more about the elements of $\mathscr{A}(R)$ which have inverses and enables us to partially describe $\mathscr{D}(R)$ in certain cases where $\mathscr{A}(R)$ may not be a group.

LEMMA 1.22. *If $[A] \in \mathscr{A}(R)$ is such that $[A]$ has an inverse then $[A] = [\tilde{A}]$.*

Proof. If $[A] \in \mathscr{A}(R)$ has an inverse then $-[A] = [R:A]$ by Proposition 1.15. Now $A \subseteq \tilde{A}$ and $R:A = R:\tilde{A}$. Thus $A(R:A) \subseteq \tilde{A}(R:A) = \tilde{A}(R:\tilde{A}) \subseteq R$. These containment relations yield the following: $0 = [A(R:A)] \geqq [\tilde{A}(R:\tilde{A})] \geqq 0$. Thus

$$0 = [A]+[R:A] = [\tilde{A}]+[R:A] \quad \text{and} \quad [A] = [\tilde{A}].$$

COROLLARY 1.23. *If $[A]$, $[B]$ have inverses in $\mathscr{A}(R)$, then $[\tilde{A}]+[\tilde{B}] = [\tilde{A}\tilde{B}] = [AB]^{\sim}$; i.e., $v(\tilde{A}\tilde{B}) = v(AB)^{\sim}$ for all $v \in F$.*

Proof. By 1.22 above, if $[A]$, $[B]$ have inverses then $[A] = [\tilde{A}]$ and $[B] = [\tilde{B}]$. Moreover, $[A]+[B]$ has an inverse. Thus $[A]+[B] = [AB] = [AB]^{\sim}$ by 1.22; i.e., $[\tilde{A}]+[\tilde{B}] = [\tilde{A}\tilde{B}] = [AB]^{\sim}$.

Now consider the map $\rho: \mathscr{D}(R) \to X^I$ defined by $\rho(\operatorname{div}_R(A)) = (v_i(\tilde{A}))_{i \in I}$. ρ is well defined, for if $\operatorname{div}_R(A) = \operatorname{div}_R(B)$ then $\tilde{A} = \tilde{B}$. Conversely, if $(v_i(\tilde{A}))_{i \in I} = (v_i(\tilde{B}))_{i \in I}$ then $[\tilde{B}] = [\tilde{A}]$ and so $\operatorname{div}_R(A) = \operatorname{div}_R(\tilde{A}) = \operatorname{div}_R(\tilde{B}) = \operatorname{div}_R(B)$ by the remark following the proof of 1.18. Thus ρ is 1-1. We can now give a description of $\mathscr{D}(R)$ when R is fairly well behaved.

THEOREM 1.24. *The map $\rho: \mathscr{D}(R) \to X^I$ defined by $\rho(\operatorname{div}_R(A)) = (v_i(\tilde{A}))_{i \in I}$ is 1-1. Furthermore ρ is a homomorphism iff $[\tilde{A}] \in \mathscr{A}(R)$ has an inverse for all $A \in I(R)$.*

Proof. The first assertion is proved in the immediately preceding remarks. We now prove the second assertion.

($\Rightarrow$) Suppose ρ is a homomorphism. Then since $\mathscr{D}(R)$ is a group, for $\operatorname{div}_R(A) \in \mathscr{D}(R)$, $-\operatorname{div}_R(A) = \operatorname{div}_R(R:A)$. Thus $\rho(\operatorname{div}_R(A) + \operatorname{div}_R(R:A)) = 0 = \rho(\operatorname{div}_R(A)) + \rho(\operatorname{div}_R(R:A)) = (v_i(\tilde{A}))_{i \in I} + (v_i(R:A))_{i \in I}$. It follows that $[\tilde{A}]$ has an inverse in $\mathscr{A}(R)$.

($\Rightarrow$) Suppose that $[\tilde{A}]$ has an inverse for all $A \in I(R)$. By Corollary 1.23, we have that $[\tilde{A}\tilde{B}] = [AB]^{\sim}$ for all $A, B \in I(R)$. Thus for $A, B \in I(R)$ we have $\rho(\operatorname{div}_R(AB)) = (v_i(AB))_{i \in I} = (v_i(\tilde{A}\tilde{B}))_{i \in I} = (v_i(\tilde{A}))_{i \in I} + (v_i(\tilde{B}))_{i \in I} = \rho(\operatorname{div}_R(A)) + \rho(\operatorname{div}_R(B))$ and ρ is a homomorphism.

Now, let R be an AK-domain. Then R_P is a Krull domain for any prime ideal P of R. However, these are not the only Krull domains T such that $R \subseteq T \subseteq K$. For if $\Delta = \{P_1, \ldots, P_n\}$ is any finite collection of prime ideals of R then $T = \bigcap_{P_i \in \Delta} R_{P_i}$ is also a Krull domain. Thus there is a large class of Krull domains T such that

$R \subseteq T \subseteq K$. When R is an AK-domain in which every minimal prime is divisoriel we always have that $\mathscr{D}(T)$ is a homomorphic image of $\mathscr{D}(R)$, where T is an AK-domain such that $R \subseteq T \subseteq K$. For, $\mathscr{A}(T)$ is a homomorphic image of the group $\mathscr{A}(R)$ and so is a group. Then $\mathscr{A}(R) \cong \mathscr{D}(R)$ and $\mathscr{A}(T) \cong \mathscr{D}(T)$. When T is a Krull domain and R is an AK-domain for which the map ρ of Theorem 1.24 is a homomorphism we also get that $\mathscr{D}(T)$ is a homomorphic image of $\mathscr{D}(R)$ as follows.

PROPOSITION 1.25. *Let R be an* AK-*domain and let T be a Krull domain such that $R \subseteq T \subseteq K$. If $[\tilde{A}]$ has an inverse for every $[A] \in \mathscr{A}(R)$ then the map $\tau: \mathscr{D}(R) \to \mathscr{D}(T)$, defined by $\tau(\operatorname{div}_R(A)) = \operatorname{div}_T(\tilde{A}T)$, is a homomorphism of $\mathscr{D}(R)$ onto $\mathscr{D}(T)$.*

Proof. τ is well defined, for if $\operatorname{div}_R(A) = \operatorname{div}_R(B)$ then $\tilde{A} = \tilde{B}$ so that $\tilde{A}T = \tilde{B}T$. Now consider the following diagram.

Diagram 1.26.

$$\begin{tikzcd} \mathscr{D}(R) \arrow[r, "\rho"] \arrow[d, "\tau"'] & Z^I \arrow[d, "\pi"] \\ \mathscr{D}(T) \arrow[r, "\gamma"] & Z^{(J)} \end{tikzcd}$$

Here, I is the index set for the family of essential valuations of R; J is the index set for the family of essential valuations of T; π is the projection of Z^I onto $Z^{(J)}$; ρ is the (injective) homomorphism of 1.24; γ is the injection of 1.4. It is well known that γ is also surjective; i.e., γ is an isomorphism. Consider the map $\gamma^{-1} \circ \pi \circ \rho$: $\mathscr{D}(R) \to \mathscr{D}(T)$. We have, for $\operatorname{div}_R(A) \in \mathscr{D}(R)$, $(\gamma^{-1} \circ \pi \circ \rho)(\operatorname{div}_R(A)) = (\gamma^{-1} \circ \pi) \times (v_i(\tilde{A}))_{i \in I} = \gamma^{-1}((v_j(\tilde{A}))_{j \in J})$. Since T is a Krull domain we have that $v(B) = v(\tilde{B})$ for all fractionary ideals B of T and all essential valuations v. Thus $(v_j(\tilde{A}))_{j \in J} = (v_j(\tilde{A}T))_{j \in J} = (v_j(\tilde{A}T)^{\sim})_{j \in J}$ so that $\gamma^{-1}((v_j(\tilde{A}))_{j \in J}) = \operatorname{div}_R(\tilde{A}T)$. Then $\gamma^{-1} \circ \pi \circ \rho = \tau$ and τ is a homomorphism since τ is a composition of homomorphisms. To see that τ is surjective it is sufficient to show that for every divisoriel fractionary ideal $\mathscr{U}$ of T there is a divisoriel fractionary ideal A of R such that $\tau(\operatorname{div}_R(A)) = \operatorname{div}_T(\mathscr{U})$. So let $\mathscr{U}$ be a fractionary ideal of T. There are elements $x, y \in K$ such that $\mathscr{U} = Tx \cap Ty$ [1, p. 13]. Let $A = Rx \cap Ry$. Then A is divisoriel and $v_j(A) = v_j(\mathscr{U})$ for all $j \in J$ and $\tau(\operatorname{div}_R(A)) = \operatorname{div}_T(\mathscr{U})$.

2. Let R be an integral domain with quotient field K. Suppose that F is a family of valuations on K satisfying the following:

(1) $R = \bigcap_{v \in F} R_v$,

(2) $R_v = R_{P(v)}$, for each $v \in F$.

Following Gilmer in [3], we make the following definition.

DEFINITION 2.1. We say that R satisfies property ($\circledast$) with respect to F iff for distinct subsets F_1, F_2 of F we have that $\bigcap_{w \in F_1} R_w \neq \bigcap_{u \in F_2} R_u$.

When R is a Prufer domain and F is the family of valuations induced by the collection of maximal ideals, then property ($\circledast$) is the same as property ($\circledast$) in [2].

For $v \in F$, we let $F_v = F - \{v\}$.

PROPOSITION 2.2. *R has property ( ) with respect to F iff for each $v \in F$, $\bigcap_{w \in F_v} R_w \nsubseteq R_v$.*

Proof. The proof is substantially the same as that of Lemma 1 in [2] and is omitted.

COROLLARY 2.3. *If R satisfies ($\circledast$) with respect to F and if G is any nonempty subset of F, then $T = \bigcap_{u \in G} R_u$ satisfies ($\circledast$) with respect to G.*

We note that if R satisfies ($\circledast$) with respect to F then $P(v) \nsubseteq P(w)$ for $v \neq w$. For if $P(v) \subseteq P(w)$ for some $w \neq v$, then $R_{P(w)} \subseteq R_{P(v)}$; i.e., $R_w \subseteq R_v$. Then we have the following: $(\bigcap_{u \in F - \{v,w\}} R_u) \cap (R_v \cap R_w) = (\bigcap_{u \in F - \{v,w\}} R_u) \cap R_w = \bigcap_{u \in F_v} R_u$, and $F \neq F_v$, a contradiction.

PROPOSITION 2.4. *If F is of finite character and is such that $P(u) \nsubseteq P(v)$ if $u \neq v$, then R satisfies ($\circledast$) with respect to F.*

Proof. Let $v \in F$ and let $x \in R$, $x \neq 0$, be such that $v(x) > 0$. Let $v_1, \ldots, v_n$ be the distinct (from v and each other) valuations such that $v_i(x) \neq 0$, $i = 1, \ldots, n$. There exists $y \in (\bigcap_{i=1}^n P(v_i)) - P(v)$. For if $\bigcap_{i=1}^n P(v_i) \subseteq P(v)$, then $\prod_{i=1}^n P(v_i) \subseteq \bigcap_{i=1}^n P(v_i) \subseteq P(v)$ and so $P(v_j) \subseteq P(v)$ for some j, $1 \leqq j \leqq n$, contradicting our hypothesis. Choose n large enough so that $w(y^m/x) \geqq 0$ for $w \in F_v$. This is possible since F is of finite character and $w(y) \geqq 0$ for all $w \in F_v$. Then $w(y^m/x) \geqq 0$ for all $w \in F_v$ and $v(y^m/x) = -v(x) < 0$. Thus $y^m/x \in (\bigcap_{w \in F_v} R_w) - R_v$; i.e., $\bigcap_{w \in F_v} R_w \nsubseteq R_v$. So R satisfies ($\circledast$) with respect to F by 2.2.

Let R be an integral domain with family F of valuations satisfying (1) and (2) listed at the beginning of this section. R is called a generalized Krull domain if F satisfies the following two additional properties (see [5]).

(3) Each $v \in F$ has rank one.

(4) F is of finite character.

COROLLARY 2.5. *If R is a Krull domain, or a generalized Krull domain with family F of valuations, then R satisfies ($\circledast$) with respect to F.*

Proof. In this case F is a family of rank one valuations of finite character, so that if $u, v \in F$, $u \neq v$, then $P(v) \nsubseteq P(u)$.

PROPOSITION 2.6. *Let R be an* AD*-domain. The following conditions on R are equivalent.*

(1) *R satisfies ($\circledast$) with respect to F, the family of essential valuations of R.*

(2) *R is Dedekind.*

(3) *Every minimal prime of R is divisoriel.*

Proof. (1) $\Leftrightarrow$ (2) is Theorem 3 of [3].

(2) $\Leftrightarrow$ (3) is found in [6].

Thus we see that in the case of almost-Dedekind domains, the divisoriel property of the minimal prime ideals completely determines whether or not R satisfies property (※). We shall see that the divisoriel property of the minimal primes is always sufficient for R to satisfy (※).

Proposition 2.7. *Let R be an integral domain with family F of valuations such that*

(i) *Each $v \in F$ has rank one.*

(ii) $R = \bigcap_{v \in F} R_v$.

(iii) $R_v = R_{P(v)}$ *for each* $v \in F$.

If $P(v)$ is divisoriel for each $v \in F$, then R satisfies (※) with respect to F.

Proof. We note that since R is the intersection of rank one valuation rings, R is completely integrally closed and hence $\mathcal{D}(R)$ is a group. If each $P(v)$ is divisoriel, then each $v \in F$ is discrete. For if $P = P(v)$ is divisoriel we must have $P^2 < P$. For if $P^2 = P$, then $\operatorname{div}_R(P^2) = \operatorname{div}_R(P)$; i.e., $2 \operatorname{div}(P) = \operatorname{div}(P)$. Thus $\operatorname{div}(P) = 0$ and $\tilde{P} = R \neq P$, contradicting $\tilde{P} = P$.

Since $P^2 < P$, we have $P^2R_P < PR_P$ and so R_P is a discrete valuation ring. We now show that $\{P(v) \mid v \in F\}$ is the set of all minimal divisoriel primes of R. Clearly, $\{P(v) \mid v \in F\}$ is contained in the set of all divisoriel minimal primes. Now let P be a minimal, divisoriel prime of R. If $P \neq P(v)$ for any $v \in F$, then $P \nsubseteq P(v)$ for any $v \in F$ and so $v(P) = 0$ for all $v \in F$; i.e., $[P] = 0$. But then we would have $g([P]) = 0$; i.e., $\operatorname{div}(P) = 0$; i.e., $\tilde{P} = R$, contradicting $\tilde{P} = P < R$. So we must have that

$$\{P(v) \mid v \in F\}$$

is the set of all divisoriel minimal primes of R. Now let G be any subset of F such that $R = \bigcap_{u \in G} R_u$. $P(u)$ is divisoriel for each $u \in G$ since $G \subseteq F$. By what we have just shown, $\{P(u) \mid u \in G\}$ is the collection of all minimal divisoriel primes of R; i.e., $G = F$. Thus for any $v \in F$, $\bigcap_{u \in F_v} R_u \nsubseteq R_v$ and so R satisfies (※) with respect to F.

The first part of the proof of Proposition 2.7 shows that if P is the center of a rank one valuation v, then v is discrete if P is divisoriel. This enables us to characterize Krull domains in the class of generalized Krull domains as follows.

Corollary 2.8. *Let R be a generalized Krull domain with family F of valuations. R is a Krull domain iff $P(v)$ is divisoriel for each $v \in F$.*

Let R be an integral domain with quotient field K and let F be a family of valuations on K satisfying conditions (1) and (2) stated at the beginning of this section. Let x be an indeterminate and let F' denote the family of valuations on $K(x)$ which are canonical extensions of elements of F. Let G denote the family of $p(x)$-adic valuations on $K(x)$, where $p(x)$ is a nonconstant irreducible polynomial in $K[x]$. Then $F' \cup G$ is a family of valuations on $K(x)$ satisfying (1) and (2) with $R[x]$ in place of R.

PROPOSITION 2.9. *If R satisfies* (※) *with respect to F, then $R[x]$ satisfies* (※) *with respect to $F' \cup G$.*

Proof. Let $w \in F' \cup G$. If $w \in G$, then w is a $p(x)$-adic valuation for some nonconstant irreducible polynomial $p(x) \in K[x]$. Without loss of generality we may assume that $p(x) \in R[x]$. Suppose $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, $a_i \in R$. Let $b = \prod_{a_k \neq 0} a_k$. Then $b \neq 0$ since $a_n \neq 0$, and $v(b) = \sum_{a_k \neq 0} v(a_k) \geqq \min_{0 \leqq j \leqq n} v(a_j)$ for all $v \in F$ since $b \in R$ and $a_k \in R$ for all $k = 0, 1, \ldots, n$, and every $v \in F$ is nonnegative on R. Then for $v' \in F'$, $v'(b/p(x)) = v'(b) - v'(p(x)) = v(b) - \min_{0 \leqq j \leqq n} v(a_j) \geqq 0$. If $u \in G$ and $u \neq w$, then u is a $q(x)$-adic valuation for some nonconstant irreducible polynomial $q(x)$ such that $q(x) \nmid p(x)$. Then $u(b/p(x)) = 0$. Thus $b/p(x) \in \bigcap_{u \in (F \cup G)_w} (R[x])_u$. $b/p(x) \notin (R[x])_w$ since $w(b/p(x)) = -1 < 0$. Thus if $w \in G$ $\bigcap_{u \in (F' \cup G)_w} (R[x])_u \nsubseteq (R[x])_w$. On the other hand, if $w \in F'$, then $w = v'$ for some $v \in F$. Since $\bigcap_{u \in F_v} R_u \nsubseteq R_v$, there is $a \in (\bigcap_{u \in F_v} R_u) - R_v \subseteq (\bigcap_{u' \in F'_{v'}} R[x]_{u'}) \cap K[x] = (\bigcap_{u' \in F'_{v'}} R[x]_{u'}) \cap (\bigcap_{z \in G} R[x]_z) = \bigcap_{w \in (F' \cup G)_{v'}} R[x]_w$, and $a \notin R_v$. Then $a \notin R[x]_{v'}$, for $v'(a) = v(a) < 0$. Thus for every $w \in F' \cup G$ we have $\bigcap_{u \in (F' \cup G)_w} R[x]_u \nsubseteq R[x]_w$ and thus $R[x]$ satisfies (※) with respect to $F' \cup G$ by 2.2.

3. In [6] it was shown that if R is an almost-Dedekind domain with family F of essential valuations, then R is Dedekind iff every minimal prime of R is divisoriel. Thus in an AD-domain R, every minimal prime is divisoriel iff F is of finite character. In §1 it was conjectured that if R is an AK-domain with family F of essential valuations, then R is Krull if $P(v)$ is divisoriel for each $v \in F$; i.e., F is of finite character if $P(v)$ is divisoriel for each $v \in F$. In this section we give an example to show that this conjecture is false if the AK requirement is dropped. We also give an example of an AK-domain which is neither a Krull domain nor an AD-domain.

Let R denote the set of entire functions, C denote the set of complex numbers, Z denote the additive group of integers. It is well known that R is an integral domain under the usual pointwise definitions of addition and multiplication. For $a \in C$ we define $v_a \colon R - \{0\} \to Z$ by $v_a(f(z)) = n$ if a is a zero of $f(z)$ of order n. If a is not a zero of $f(z)$ then $v_a(f(z)) = 0$. If $f(z) \equiv 0$ we put $v_a(f(z)) = +\infty$ for each $z \in C$. It is easy to show that each v_a can be extended to a valuation on the quotient field of R. We let F denote this family of valuations. F has the following properties: (i) Each $v \in F$ has rank one and is discrete; (ii) $R = \bigcap \{R_v \mid v \in F\}$; (iii) $R_v = R_{P(v)}$ for each $v \in F$; (iv) For $a \in C$, $P(v_a) = (z-a)R$, and hence is divisoriel; (v) F is not of finite character. Furthermore, $P(v)$ is maximal for each $v \in F$. However, these are not all the maximal ideals of R. For let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers such that $\lim z_n = \infty$. For each positive integer m, let $f_m(z)$ be an entire function whose zeros are exactly $\{z_m, z_{m+1}, \ldots\}$. The ideal generated by $\{f_1(z), f_2(z), \ldots\}$ is proper and is contained in a maximal ideal M. However, R_M is not a Krull domain. It follows that R is not AK.

It was shown in [7] that if R is AK and $X_1, \ldots, X_n$ are indeterminates, then $R[X_1, \ldots, X_n]$ is AK. Let R be an AD-domain which is not a Dedekind domain.

Such a domain is given in example 2 of [2]. Then $R[X_1, \ldots, X_n]$ is an AK-domain which is neither a Krull domain nor an AD-domain. We observe that example 1 of [2] is a generalized Krull domain which is not a Krull domain.

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